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Direct Computation of Orbital Sunrise or Sunset Event Parameters

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INTRODUCTION

A number of upcoming space projects - for example, Langley Research Center's SAGE II and HALOE projects - make use of instruments carried aboard Earth orbital satellites which view the Sun and measure the intensity of solar radiation as the Sun "rises" or "sets" with respect to the spacecraft. The atmospheric optical path through which the spacecraft views the Sun varies in length during such an event and attenuates the measured solar radiation to varying degrees. Consequently, by the use of established inversion techniques, the vertical distribution of optically active constituents along the ray path can be mapped.

A number of geometric parameters defined by the relative positions of the spacecraft, the Earth, and the Sun are used both in mission planning and during mission operations and data reduction. Many of these parameters are highly dependent on the orientation of the orbital plane with the Earth-Sun direction.

In the mission planning stages, for example, a number of these parameters are used to determine the maximum required excursion of instrument pointing angles - elevation and azimuth angles - and their rates of change for proper sizing and placement of structures such as gimbals and drive mechanisms and for imposing accuracy requirements on the design performance of these structures. The tangent height, defined as the minimum altitude above the Earth's surface of the line of sight from the spacecraft to some point of the solar disk, is also an important parameter. In addition to determining the geographic "location" of the measurement, it is used to define the mission event time, which is the time required to pass between two given tangent heights. This time determines the data collection period, which in turn can set requirements on data collection rates to insure that sufficient data are taken during the event. This time may also affect the thermal design of the instrument or its peripheral support components, as it determines the total time the instrument hardware might be exposed to the direct rays of the Sun.

Traditionally, for elliptic orbits over an oblate Earth, these parameters are computed by dividing the spacecraft orbit into several hundred steps, computing these parameters at each step, and then interpolating in the resultant tables for the required values. This procedure can be rather expensive in terms of computer time, as over a typical 3- or 4-year mission, it may be necessary to compute thousands of orbits to determine the maximum design ranges of these instrument parameters and to predict the time and location of future events. For mission design purposes, however, the analysis can be greatly simplified if one is willing to assume that only the secular perturbations produced by the oblate gravity field are important and that the short-period and long-period perturbations, which are generally an order of magnitude or more smaller than the secular perturbations, can be neglected. Using these assumptions, Brooks (1977) presents a concise but thorough review of orbit mechanics from a mission-planning point of view. Brooks (1980) applies these results to the design of solar occultation missions. This paper concentrates on a detailed examination of the rise or set problem, which is the operational mode of many Earth satellite experiments and hence complements the material presented in the two papers by Brooks. The inclusion of the appendix in the present paper eliminates the need for reliance on external data such as almanacs or ephemerides.

SYMBOLS

A_z	azimuth angle, deg
a	semimajor axis of orbit, km
E	eccentric anomaly, deg
E_ℓ	elevation angle above or below local horizontal, deg
e	eccentricity
$\hat{e}_r$	unit vector along radius of spacecraft
$\hat{e}_s$	unit vector to some part of solar disk
$\hat{e}_1$	unit vector to the Sun
$\hat{e}_2$	unit vector in orbital plane
$\hat{e}_3$	defined by equation (13)
GMT	Greenwich mean time
i	inclination of orbital plane to Earth equator, deg
M	mean anomaly, deg
n	mean angular rate of spacecraft in its orbit, rad/sec
$\hat{P}$	unit vector in orbital plane pointing toward periapsis
p,q,w	direction cosines of $\hat{e}_s$ vector along $\hat{P}$, $\hat{Q}$, and $\hat{W}$, respectively
$\hat{Q}$	unit vector in orbital plane pointing toward a true anomaly of 90°
R	magnitude of spacecraft radius vector, km
R_e	radius of the Earth, km
$\vec{S}_t$	vector which forms a triangle with $\vec{R}_t$ and $\vec{\rho}_t$, km
STG	sidereal time at Greenwich
STGO	Greenwich sidereal time at 0 hours GMT
T	last time of periapsis passage
t	time
v	true anomaly, deg
$\hat{W}$	unit vector normal to plane of orbit along positive direction of angular momentum vector

X_0 component of radius vector along $\hat{P}$, km

Y_0 component of radius vector along $\hat{Q}$, km

α_s right ascension of the Sun, deg

α_t right ascension of ρ_t, deg

β angle between Sun vector and orbital plane, deg

γ angular arc traced by subtangent point during rise or set event, deg

δ_s declination of the Sun, deg

δ_t declination of ρ_t, deg

ε obliquity of the ecliptic, deg

θ angle between radius vector and Sun vector, deg

λ_t local longitude (positive east), deg

ρ magnitude of projection of $\vec{R}$ onto $\hat{e}_2$-$\hat{e}_3$ plane, km

σ_x component of $\hat{e}_s$ in $-\hat{W}$ direction

σ_y component of $\hat{e}_s$ in $\hat{W} \times \hat{e}_r$ direction

Ω longitude of ascending node, deg

ω argument of periapsis, deg

ω_s angular rate of the Earth about the Sun, rad/sec

Subscripts:

max maximum

min minimum

t sunrise or sunset event

Superscripts:

h hours

m minutes

s seconds

A dot (•) over a symbol denotes a derivative with respect to time; a caret (^) over a symbol denotes a unit vector; and an arrow (→) over a symbol denotes a dimensioned vector.

This paper presents an analytical method for computing the parameters pertinent to sunrise or sunset events (hereafter called events). No interpolation is involved, and only the parameters of interest are computed at the time and altitude conditions imposed by mission requirements. Two assumptions are made (1) the Earth is spherical, and (2) refraction is neglected. Both of these assumptions, however, can be removed by iterating on the solutions presented herein. A general elliptical orbit is assumed initially. However, once the basic equations are derived, a restriction to circular orbits is introduced, which greatly simplifies the resulting equations and permits a ready interpretation for extension to the more complex elliptic orbit geometry.

ANALYSIS

One of the many significant parameters used in mission design studies and data reduction analysis is the ray tangent height, which is the minimum altitude above the Earth's surface of the ray from the instrument optical center to some point on the solar disk, usually the center of the visible disk. The data event time is defined as the time interval required for the spacecraft to pass between two specified tangent heights. For example, on a sunset event, the data collection period may extend from an altitude of about 100 km to some small negative altitude. (Refraction being neglected, the negative geometric altitude would correspond to a ray just grazing the Earth's surface.)

The duration of the event depends strongly on the orientation of the orbital plane of the spacecraft relative to the Earth-Sun line. For example, for a given set of tangent point altitudes, the event time will be a minimum when the Sun views the orbit "edge-on."

The orbital plane, however, rotates relative to the Sun because of two different effects, one dynamic in origin and the other gravitational. If the density of the Earth were a function only of the distance from the center of the Earth, the orbital plane would remain essentially fixed in inertial space. Thus, as the Earth revolves about the Sun in its annual excursion, the orbital plane would appear to rotate relative to a solar observer. Because the Earth is not really a perfect sphere gravitationally, an inertial rotation of the orbit plane results, which is superimposed on the annual mean motion.

One consequence of this motion is that depending on the size of the orbit, the inclination of the orbital plane to the Earth's equator, and the declination of the Sun, there may come a time when a solar observer would see the orbit just graze the edge of the Earth's projected disk. Thus, to an observer on the spacecraft, the Sun would neither "rise" nor "set," but would appear to sink toward the horizon, just graze the Earth's surface, and begin to rise again. In fact, depending on the orbital geometry, a condition could arise in which the orbital plane is normal to the Earth-Sun line, and hence, for a circular orbit, the Sun would appear stationary to an observer on a spacecraft.

The fundamental parameter on which these conditions are related is the so-called "beta angle" (β), which is the angle the Earth-Sun vector makes with its projection onto the orbital plane of the spacecraft (fig. 1).

The following development assumes a general elliptic orbit. However, although the principle is exactly the same, much of the algebra becomes complicated very rapidly and tends to obscure the basic dependence on β. Thus, once the general equations are developed, a restriction to circular orbits will be imposed. The simplified equations which result permit a ready physical interpretation which can easily be extrapolated to the more complex case of elliptic orbits.

The basic approach to be taken is to project the radius vector of the spacecraft onto a plane normal to the Earth-Sun vector and view the orbit as seen by an observer on the Sun. The minimum projected radius of the spacecraft as thus seen, which is directly related to β, is then used to determine whether or not event conditions exist, and if so, to determine the event times, elevation and azimuth angles, and other parameters of use to the mission planner.

Figure 2 illustrates the geometry. The fundamental plane defined in figure 2 is the plane of the spacecraft orbit, with pole $\hat{W}$. The vector $\hat{e}_s$ is a unit vector to some part of the solar disk. (The center of the disk is assumed here.) It is tacitly assumed that the Sun is at an infinite distance. With very little additional complexity, the coordinate origin could be transferred to the spacecraft, and a finite distance assumed for the Sun. A slight correction to the solar right ascension and declination would thereby be introduced to give these quantities as seen from the spacecraft coordinate system. These small parallax effects are neglected in the present text. If the right ascension and declination of the Sun are given by α_s and δ_s for a specific date and time (see the appendix), then

$$\hat{e}_s = \begin{bmatrix} \cos \delta_s \quad \cos \alpha_s \\ \cos \delta_s \quad \sin \alpha_s \\ \sin \delta_s \end{bmatrix} \tag{1}$$

Symbols $\hat{P}$ and $\hat{Q}$ represent unit vectors in the orbital plane, with $\hat{P}$ pointing toward periapsis and $\hat{Q}$ pointing toward a true anomaly of 90°. The symbol $\hat{W}$ represents a unit vector normal to the plane of the orbit along the positive direction of the angular momentum vector. In terms of the usual orbital elements (e.g., see Escobal 1965),

$$\hat{P} = \begin{bmatrix} \cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i \\ \cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i \\ \sin \omega \sin i \end{bmatrix} \tag{2}$$

$$\hat{Q} = \begin{bmatrix} -\sin \omega \cos \Omega - \cos \omega \sin \Omega \cos i \\ -\sin \omega \sin \Omega + \cos \omega \cos \Omega \cos i \\ \cos \omega \sin i \end{bmatrix} \tag{3}$$

$$\hat{W} = \begin{bmatrix} \sin \Omega \sin i \\ -\cos \Omega \sin i \\ \cos i \end{bmatrix} \tag{4}$$

where ω is the argument of periapsis, Ω is the longitude of the ascending node, and i is the inclination of the orbital plane to the Earth equator.

The spacecraft radius vector $\vec{R}$ is given by (Escobal 1965)

$$\vec{R} = X_0\hat{P} + Y_0\hat{Q} \tag{5}$$

where

$$X_0 = a(\cos E - e) \tag{6}$$

and

$$Y_0 = a\sqrt{1 - e^2} \sin E \tag{7}$$

In equations (6) and (7), a is the semimajor axis of the orbit, e is the eccentricity, and E is the eccentric anomaly, related to the true anomaly v and the mean anomaly M by the equations

$$\tan \frac{E}{2} = \sqrt{\frac{1 - e}{1 + e}} \tan \frac{v}{2} \tag{8}$$

$$M = n(t - T) = E - e \sin E \tag{9}$$

The last equation relates E to time t, where T is the last time of periapsis passage, and n is the mean angular rate of the spacecraft in its orbit

$$n = \sqrt{\frac{\mu}{a^3}} \tag{10}$$

In equation (10), μ is the gravitational constant of the central planet, which for the Earth is 398 600.64 km^3/sec^2.

Define the orthogonal vector triad

$$\hat{e}_1 = \hat{e}_s \tag{11}$$

$$\hat{e}_2 = \frac{\hat{W} \times \hat{e}_s}{\left| \hat{W} \times \hat{e}_s \right|} \tag{12}$$

$$\hat{e}_3 = \hat{e}_1 \times \hat{e}_2 \tag{13}$$

where $\hat{e}_1$ is a unit vector to the Sun, and $\hat{e}_2$ is a unit vector in the orbital plane.

From the geometry of figure 2, we see that

$$\sin \beta = \hat{W} \cdot \hat{e}_s \tag{14}$$

$$\cos \beta = \left| \hat{W} \times \hat{e}_s \right| \tag{15}$$

where β is the angle between the Sun vector and the orbital plane. Thus, from equation (12), we can write

$$\hat{e}_2 = \frac{\hat{W} \times \hat{e}_s}{\cos \beta} \tag{16}$$

and by use of the vector identity (e.g., Kaplan 1973)

$$(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{C} \cdot \vec{A})\vec{B} - (\vec{C} \cdot \vec{B})\vec{A}$$

we can write $\hat{e}_3$ as

$$\hat{e}_3 = \frac{\hat{e}_s \sin \beta - \hat{W}}{\cos \beta} \tag{17}$$

Let R_2 and R_3 denote the projections of $\vec{R}$ onto $\hat{e}_2$ and $\hat{e}_3$, respectively. Then

$$R_2 = \vec{R} \cdot \hat{e}_2 = \frac{\vec{R} \cdot (\hat{W} \times \hat{e}_s)}{\cos \beta}$$

and using another vector identity from Kaplan (1973)

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{C} \times \vec{A}) \cdot \vec{B}$$

we get

$$R_2 = \frac{(\vec{R} \times \hat{W}) \cdot \hat{e}_s}{\cos \beta}$$

From figure 2

$$\hat{P} \times \hat{W} = -\hat{Q}$$

$$\hat{Q} \times \hat{W} = \hat{P}$$

and hence, using equation (5), we obtain

$$R_2 = \frac{-X_0(\hat{Q} \cdot \hat{e}_s) + Y_0(\hat{P} \cdot \hat{e}_s)}{\cos \beta} \tag{18}$$

Similarly, since $\hat{R} \cdot \hat{W} = 0$ by construction, we get

$$R_3 = \left[X_0(\hat{P} \cdot \hat{e}_s) + Y_0(\hat{Q} \cdot \hat{e}_s)\right] \tan \beta \tag{19}$$

Define ρ, the magnitude of the projection of $\vec{R}$ onto the $\hat{e}_2$-$\hat{e}_3$ plane by

$$\rho^2 = R_2^2 + R_3^2$$

which, with the direction cosines of the $\hat{e}_s$ vector

$$p = \hat{P} \cdot \hat{e}_s$$

$$q = \hat{Q} \cdot \hat{e}_s$$

$$w = \hat{W} \cdot \hat{e}_s$$

can be written as

$$\rho^2 = R^2 \sin^2 \beta + (X_0 q - Y_0 p)^2 \tag{20}$$

$$\rho^2 = R^2 - (X_0 p + Y_0 q)^2 \tag{21}$$

where

$$R = \sqrt{X_0^2 + Y_0^2} = a(1 - e \cos E) \tag{22}$$

and use has been made of the direction cosine property

$$p^2 + q^2 + w^2 = 1 \tag{23}$$

Given a particular value of ρ corresponding to an event altitude (tangent point height, for example), equation (20) or (21) can be solved by substitution from equations (6) and (7) for E, the eccentric anomaly for which the event occurs. A second event condition would give a second value of E, and consequently the time interval between the two events can be determined from equation (9) as follows:

$$\delta t = \frac{(E_2 - E_1) - e(\sin E_2 - \sin E_1)}{n} \tag{24}$$

Equation (20) or (21) is in general a quartic equation for $\cos E$ (or $\sin E$), and hence there are eight roots to be investigated, since the sign of the sine and cosine functions only determines one of two quadrants into which E can fall. However, only four of the eight values of E will yield the correct value of ρ, and hence equation (20) must be used to determine which four roots correspond to the physically real roots. (See fig. 3 and the numerical example presented later.)

Some other parameters of considerable importance in mission design studies are the elevation and azimuth angles of the Sun as seen in a local spacecraft coordinate frame and the time rates of change of these angles.

Form two new unit vectors: (1) $\hat{e}_r = \vec{R}/|\vec{R}|$, a unit vector along the radius vector of the spacecraft, and (2) $\hat{W} \times \hat{e}_r$ (fig. 4). For circular orbits, the velocity is parallel to $\hat{W} \times \hat{e}_r$. Thus, the triad $-\hat{W}$, $\hat{W} \times \hat{e}_r$, and $\hat{e}_r$ form a useful local coordinate system, with $\hat{W} \times \hat{e}_r$ pointing "forward," $-\hat{W}$ pointing out the "right wing," and $\hat{e}_r$ pointing "upward" along the positive radius vector, the local vertical. (See fig. 5.)

In this coordinate system, centered at the spacecraft, any vector, specifically $\hat{e}_s$, can be defined by E_ℓ, the elevation angle above or below the local horizontal (the $-\hat{W}$, $\hat{W} \times \hat{e}_s$ plane), and A_z, the azimuth angle measured clockwise from, say, the $\hat{W} \times \hat{e}_r$ vector in the local horizontal plane.

From figure 5 and equations (5) and (21), the elevation angle is

$$\sin E_\ell = \hat{e}_r \cdot \hat{e}_s = -\sqrt{1 - \left(\frac{\rho}{R}\right)^2} \tag{25}$$

The minus sign is chosen by convention to make the elevation angle negative when it is measured below the local horizontal.

Let σ_x and σ_y be the components of $\hat{e}_s$ in the $-\hat{W}$ and $\hat{W} \times \hat{e}_r$ directions, respectively. Then,

$$\sigma_x = -\hat{e}_s \cdot \hat{W} \tag{26a}$$

and

$$\sigma_y = \hat{e}_s \cdot \left(\hat{W} \times \hat{e}_r\right) = -\left(\hat{e}_r \cdot \hat{e}_2\right) \cos \beta \tag{26b}$$

From equations (14) and (15), σ_x and σ_y can be written as

$$\sigma_x = -\sin \beta \tag{27}$$

$$\sigma_y = \frac{qX_0 - pY_0}{R} \tag{28}$$

Thus the azimuth angle and its quadrant can be determined from

$$\sin A_z = \frac{\sigma_x}{\sqrt{\sigma_x^2 + \sigma_y^2}} \tag{29}$$

$$\cos A_z = \frac{\sigma_y}{\sqrt{\sigma_x^2 + \sigma_y^2}} \tag{30}$$

From figure 5, it can be seen that since $\hat{e}_3$ is a unit vector,

$$\sqrt{\sigma_x^2 + \sigma_y^2} = \cos E_\ell \tag{31}$$

and so equations (29) and (30) appear as

$$\sin A_z = \frac{-\sin \beta}{\cos E_\ell} \tag{32}$$

$$\cos A_z = \frac{qX_0 - pY_0}{R \cos E_\ell} \tag{33}$$

It is of interest to determine the geographic position of the subtangent point at the event time. Let the subscript t refer to either a sunrise or a sunset event. The position vector of the spacecraft at the event time is found from equations (5) through (7) with the appropriate value of E used in equations (6) and (7). Figure (6) shows the tangent point geometry, where $\vec{\rho}_t$ is the tangent point vector, and $\vec{S}_t$ is a vector which forms a triangle with $\vec{R}_t$ and $\vec{\rho}_t$ points toward the center of the Sun.

The vector $\vec{S}_t$ can be written as

$$\begin{aligned}\vec{S}_t &= \hat{e}_s R_t \cos \theta \\ &= -R_t(\hat{e}_r \cdot \hat{e}_s)\hat{e}_s\end{aligned}$$

where θ is the angle between the radius vector and the Sun vector (see fig. 6), so that

$$\vec{\rho}_t = \vec{R}_t - R_t(\hat{e}_r \cdot \hat{e}_s)\hat{e}_s \tag{34}$$

The right ascension α_t and declination δ_t of $\vec{\rho}_t$ can be found from

$$\left.\begin{aligned} \rho_x &= \rho_t \cos \delta_t \cos \alpha_t \\ \rho_y &= \rho_t \cos \delta_t \sin \alpha_t \\ \rho_z &= \rho_t \sin \delta_t \end{aligned}\right\} \tag{35}$$

The geocentric latitude of the subtangent point is identical with the declination. To get the longitude, we must compute the sidereal time at the event time.

The orbital elements used in all the previous calculations are assumed to be given at a specified time, t (GMT) and are given with respect to an inertial coordinate system in which the x-y plane is the Earth equatorial plane, and the x-axis is defined by the direction of the vernal equinox. (See Smart 1977, Brooks 1977, or Escobal 1965.) The sidereal time, as applied here, locates the Greenwich meridian in this coordinate system (see fig. 7) from which we get the relation with reference to the subtangent point

$$\alpha_t = \text{STG} + \lambda_t \tag{36}$$

where STG is the sidereal time at Greenwich and λ_t is the local longitude (positive east). The Greenwich sidereal time at 0 hours GMT (STGO) on the date is found from the equations in the appendix. If the orbital elements are specified at time t (GMT) and if Δt is the time of the event measured from t, then the sidereal time of the event is given by

$$\text{STG} = \text{STGO} + 0.25068447\ \Delta t \tag{37}$$

where Δt is given in minutes from 0 hours GMT, and the constant is the rotational rate of the Earth in degrees per mean solar minute. The latitude and longitude of the subspacecraft point can be found in a similar manner by using $\vec{R}_t$ in place of $\vec{\rho}_t$ above.

Simplification for Circular Orbits

The imposition of circular orbits permits a great simplification in the evaluation of the above equations. First, one can set $\omega = 0$, since periapsis is undefined for a circular orbit, and measure E from the ascending node. This permits the equations for $\hat{P}$ and $\hat{Q}$ to be written as

$$\hat{P} = \begin{bmatrix} \cos \Omega \\ \sin \Omega \\ 0 \end{bmatrix} \tag{38}$$

$$\hat{Q} = \begin{bmatrix} -\sin \Omega \cos i \\ \cos \Omega \cos i \\ \sin i \end{bmatrix} \tag{39}$$

and equations (6) and (7) reduce to

$$X_0 = R \cos E \tag{40}$$

$$Y_0 = R \sin E \tag{41}$$

where now R is a constant, and $E = M$. Equations (20) and (21) reduce to

$$\left.\begin{aligned} \rho^2 &= R^2 \sin^2 \beta + R^2 (q \cos E - p \sin E)^2 \\ &= R^2 [1 - (p \cos E + q \sin E)^2] \end{aligned}\right\} \tag{42}$$

The minimum value of ρ is found from equation (42) to be

$$\rho_{min} = R|\sin \beta| \tag{43}$$

which is the semiminor diameter of the elliptical projection of the circular orbit onto the $\hat{e}_2$-$\hat{e}_3$ plane. If $\rho_{min} < R_e$, the radius of the Earth, there will be a pair of distinct sunrise and sunset events. If $\rho_{min} = R_e$, the orbit will just graze the surface of the Earth as seen by a solar observer, and the sunrise and sunset roots coalesce into one value. Finally, for $\rho_{min} > R_e$, there will be no sunrise or sunset events.

If we restrict our attention now to the situation which produces a pair of events, there will be four physically meaningful roots of equations (42), as stated earlier, corresponding to the four points where the spacecraft pierces the cylinder circumscribing the Earth as seen from the Sun and as seen in figure 3. Two of these points (roots 1 and 2) correspond to the actual sunrise and sunset conditions, respectively, and pierce the "shadow cylinder" of the "shadow side" of the Earth; the other two roots (3 and 4) correspond to conditions at the same value of ρ but pierce the shadow cylinder of the "Sun side" of the Earth. These roots can be distinguished as follows: $\vec{R} \cdot \hat{e}_s$ is the component of $\vec{R}$ lying along the Earth-Sun vector. If $\vec{R} \cdot \hat{e}_s > 0$, the spacecraft is on the Sun side of the Earth (i.e., sunward of the terminator), whereas if $\vec{R} \cdot \hat{e}_s < 0$, the spacecraft is on the shadow side (i.e., behind the terminator). Thus, the real events occur for the two values of E for which

$$\vec{R} \cdot \hat{e}_s < 0 \tag{44}$$

From equation (5), this expression becomes

$$X_0 p + Y_0 q < 0$$

and for circular orbits

$$p \cos E + q \sin E < 0 \tag{45}$$

The distinction between sunrise and sunset can most readily be made by examining the sign of $\dot{\rho}$. From equations (42)

$$\dot{\rho} = \frac{-nR^2}{\rho}\left(pq \cos 2E + \frac{q^2 - p^2}{2} \sin 2E\right) \tag{46}$$

A positive $\dot{\rho}$ value identifies a sunrise event, and a negative $\dot{\rho}$ value corresponds to a sunset event.

The values of E which give a specific ρ can be found by solving equations (42) to obtain

$$\cos E = \frac{-ph \pm q\sqrt{p^2 + q^2 - h^2}}{p^2 + q^2} \tag{47}$$

where, taking into account the condition shown in equation (45), h is the positive constant

$$h = +\sqrt{1 - \left(\frac{\rho}{R}\right)^2} \tag{48}$$

Equation (47) gives two roots for $\cos E$ and hence four roots for E. Two of the four roots can be eliminated by using equation (42), as they will not produce the correct value for ρ, and the sunrise and sunset roots can be identified from equation (46).

If we write out the expression for $\sin \beta$ from equation (14) by using equations (1) and (4),

$$\sin \beta = \cos i \sin \delta_s + \sin i \cos \delta_s \sin (\Omega - \alpha_s) \tag{49}$$

For a given orbit, β is a maximum when the orbit plane is as nearly normal to the Sun vector as it can get, that is, when $\Omega - \alpha_s = 90°$ or $270°$. From equation (49) this condition corresponds to

$$\beta_{max} = \pm(i + \delta_s) \tag{50}$$

For a grazing condition, $\rho_{min} = R_e$, and we have from equation (43) that

$$\sin \beta_G = \frac{R_e}{R} \tag{51}$$

where β_G corresponds to the value of β for a grazing condition. Thus if $\beta_{max} < \beta_G$, the orbit will not experience a grazing situation - there must be events for that orbit. Of course, δ_s changes throughout the year from -23.5° to 23.5°, and hence there may be grazing possibilities at some times of the year and not at others. A grazing condition is most likely to occur near the solstices, and hence the maximum inclination an orbit can have in order to insure that it never experiences a grazing situation can be found from equations (50) and (51) if we set $\delta_s = 23.5°$. The following table reflects these vaules:

Orbit altitude, km	β, deg	i_{max}, deg
300	72.76	49.32
400	70.22	46.98
500	68.02	44.58
600	66.07	42.63
700	64.30	40.86
800	62.69	39.25
900	61.60	37.76
1000	59.82	36.38

These calculations show that if $i < i_{max}$, then no grazing condition can be experienced for that orbit, and each orbit will experience a pair of events. However, if $i > i_{max}$, there is no guarantee that a grazing situation will be experienced, as the grazing condition depends on the angles Ω, δ_s, and α_s, all of which vary throughout the year. However, since Ω and α_s both change more rapid than δ_s, especially near the solstices, the probability is high that a grazing event will occur.

The expressions for the elevation and azimuth angles assume simple forms for circular orbits. For these orbits, the unit radius vector becomes

$$\hat{e}_r = \hat{P} \cos E + \hat{Q} \sin E \tag{52}$$

and hence equation (25) can be written

$$\sin E_\ell = p \cos E + q \sin E = -\sqrt{1 - \left(\frac{\rho}{R}\right)^2} \tag{53}$$

Similarly equation (28) reduces to

$$\sigma_y = q \cos E - p \sin E \tag{54}$$

Then, using equation (53),

$$\sqrt{\sigma_x^2 + \sigma_y^2} = \sqrt{1 - (p \cos E + q \sin E)^2} = \cos E_\ell \tag{55}$$

a result we found earlier from geometric considerations, and the azimuth follows from equations (29) and (30)

$$\left.\begin{aligned} \sin A_z &= \frac{-\sin \beta}{\cos E_\ell} \\ \cos A_z &= \frac{q \cos E - p \sin E}{\cos E_\ell} \end{aligned}\right\} \tag{56}$$

The elevation and azimuth angle rates can also be computed from equation (53)

$$\frac{dE_\ell}{dt} = \frac{n(q \cos E - p \sin E)}{\cos E_\ell} = n \cos A_z \tag{57}$$

Comparing equations (55) and (42), we see that

$$\sqrt{\sigma_x^2 + \sigma_y^2} = \frac{\rho}{R} = \cos E_\ell \tag{58}$$

and hence

$$\sin A_z = \frac{-R \sin \beta}{\rho} = - \frac{\rho_{min}}{\rho} \tag{59}$$

from which

$$\frac{dA_z}{dt} = \frac{-\dot{\rho}\rho_{min}}{\rho^2 \cos A_z} \tag{60}$$

Finally, it may be of some interest to compute the rate of change of β. From equation (14)

$$\cos \beta \frac{d\beta}{dt} = \dot{\hat{W}} \cdot \hat{e}_s + \hat{W} \cdot \dot{\hat{e}}_s \tag{61}$$

If ω_s is the angular rate of the Earth about the Sun, then $\dot{\hat{e}}_s$ can be written

$$\dot{\hat{e}}_s = \omega_s \begin{bmatrix} -\cos \varepsilon \cos \delta_s \sin \alpha_s - \sin \varepsilon \sin \delta_s \\ \cos \varepsilon \cos \delta_s \cos \alpha_s \\ \sin \varepsilon \cos \delta_s \cos \alpha_s \end{bmatrix} \tag{62}$$

where ε is the obliquity of the ecliptic. (See the appendix.) With equations (1) and (4), equation (61) can be put into the form

$$\cos\beta\,\frac{d\beta}{dt} = -\omega_s[\sin i\cos\varepsilon\cos\delta_s\cos(\Omega - \alpha_s)$$

$$+ \sin\varepsilon\,(\sin\Omega\sin i\sin\delta_s - \cos i\cos\delta_s\cos\alpha_s)]$$

$$+ \dot{\Omega}\sin i\cos\delta_s\cos(\Omega - \alpha_s) \tag{63}$$

The bracketed term in equation (63) results from the annual excursion of the Earth around the Sun, and the second term results from the secular perturbation in the line of nodes. The angular rate ω_s (in degrees per day) can be well approximated by the expression

$$\omega_s \approx 1.00288 + 0.03352\cos[0.98562(d - 2)] \tag{64}$$

where d is the day number of the year, and $\dot{\Omega}$ can be found from the perturbation equations (Brooks 1977 or Escobal 1965). For circular orbits $\dot{\Omega}$ can be written as

$$\dot{\Omega} = -\frac{3}{2} J_2\left(\frac{R_e}{a}\right)^2 (\cos i)n$$

in which $J = 1.08228 \times 10^{-3}$ is the second-order perturbation constant, and $\dot{\Omega}$ has the same units as n.

One final parameter which is of frequent interest might be termed the "skewness" of the data profile - this is, the length of the angular arc traced by the subtangent point during a sunrise or sunset event. If we let this angle be γ, then

$$\cos\gamma = \frac{\vec{\rho}_{t_1} \cdot \vec{\rho}_{t_2}}{\left|\vec{\rho}_{t_1}\right|\left|\vec{\rho}_{t_2}\right|} \tag{65}$$

where the subscripts 1 and 2 refer to the beginning and end of the event, respectively. Substitute from equation (34), use equation (25), and simplify to obtain

$$\cos\gamma = \frac{\hat{e}_{r_1} \cdot \hat{e}_{r_2} - \sin E_{\ell_2}\sin E_{\ell_1}}{\cos E_{\ell_2}\cos E_{\ell_1}} \tag{66}$$

Equation (52) permits this to be written as

$$\cos \gamma = \frac{\cos(E_2 - E_1) - \sin E_{\ell_2} \sin E_{\ell_1}}{\cos E_{\ell_2} \cos E_{\ell_1}} \tag{67}$$

This arc length, when expressed in units of kilometers on the surface, can range from 0 km for $\beta = 0°$ to several hundred kilometers for $\beta = \beta_{max}$.

Numerical Examples

Some numerical examples illustrating the computations and some applications might clarify the concepts introduced in the text. One complete calculation of a pair of events is presented, and then the application of these methods to a typical ground-truth experiment is given.

Calculation of events.- We assume a set of orbital elements presented below:

$a = 6981.2908$ km

$e = 0$

$i = 57.0°$

$\Omega = 266.1083°$

$\omega = 52.5800°$

$M = 172.3795°$

Date = November 12, 1985

T = 0 hours, GMT

From the equations of the appendix, we find for this date, $\alpha_s = 227.0949°$ and $\delta_s = -17.6197°$. The sidereal time at Greenwich is 51.0702°. We assume that sunrise or sunset occurs when the tangent ray altitude is -70 km to approximate the effect of refraction of the Earth's atmosphere.

From equations (2)-(4) and (1), we get the unit vectors

$$\hat{P} = \begin{bmatrix} 0.39031 \\ -0.63561 \\ 0.66607 \end{bmatrix}$$

$$\hat{Q} = \begin{bmatrix} 0.38409 \\ 0.76991 \\ 0.50962 \end{bmatrix}$$

$$\hat{W} = \begin{bmatrix} -0.83674 \\ 0.05692 \\ 0.54464 \end{bmatrix}$$

$$\hat{e}_s = \begin{bmatrix} -0.64885 \\ -0.69812 \\ -0.30270 \end{bmatrix}$$

and the p, q, and w parameters become

$$p = -0.01114$$

$$q = -0.94097$$

$$w = 0.33832$$

Hence, β and ρ_{min} are

$$\beta = 19.77430°$$

$$\rho_{min} = 2361.882 \text{ km}$$

Since ρ_{min} is less than the radius of the Earth (6378 km), there are distinct events for this orbit. With $\rho = 6308$ km, we get from equations (47), (42), and (46), the following table of events for E:

Root	E	ρ, km	$\dot{\rho}$, km/sec
1	26.407	6308.0	-3.002
2	333.593	6372.2	
3	152.236	6308.0	3.002
4	207.764	6240.9	

and therefore roots 1 and 3 are the physically real roots, with root 1 corresponding to sunset (negative $\dot{\rho}$) and root 3 being the sunrise root. The mean angular rate of the spacecraft in its orbit is (from eq. (10)) equal to 0.06201 deg/sec. Thus, the rise and set times are

$$t_{rise} = \frac{360 - 172.3795 + 152.2363}{0.06201} = 5480.68 \text{ sec}$$

$$= 1^h\ 31^m\ 21^s$$

$$t_{set} = \frac{360 - 172.3795 + 26.4067}{0.06201} = 3451.49 \text{ sec}$$

$$= 0^h\ 57^m\ 31^s$$

At sunrise, we find from equations (5)-(7) that

$$\vec{R}_t = \begin{bmatrix} -1162.106 \\ 6430.326 \\ -2457.405 \end{bmatrix}$$

and hence the tangent point vector is (from eq. (34))

$$\vec{\rho}_t = \begin{bmatrix} -3102.970 \\ 4342.080 \\ -3362.850 \end{bmatrix}$$

The right ascension and declination of the subtangent point vector are (from eq. (35))

$$\alpha_t = 125.551°$$

$$\delta_t = -32.216°$$

Now, $1^h\ 31^m\ 21^s$ = 91.35 minutes, and thus the Greenwich sidereal time at sunrise is (see appendix)

$$STG = 51.0702 + (0.25068447)(91.35) = 73.9702°$$

and hence the longitude of the subtangent point is (from eq. (36))

$$\lambda_t = 125.551 - 73.970 = 51.581° \text{ east longitude}$$

If we repeat this calculation using the sunset root, we get the table below:

Parameter	Sunrise	Sunset
Time	$1^h\ 31^m\ 21^s$	$0^h\ 57^m\ 31^s$
Latitude	-32.216°	43.910°
Longitude	51.583°	247.614°

Now, we look at the events at an altitude of 137 km. The range of 137 to -70 km (or vice versa) represents a typical altitude range of a Sun-scan-type experiment. It is assumed that the Sun's position remains fixed and that the orbital elements retain the values they had at time T. By iterating on the results given here, these restrictions can obviously be relaxed if higher accuracy is warranted. The calculation gives E at sunrise as 156.873° and E at sunset as 21.770°. The sunrise and sunset times are $1^h\ 32^m\ 35^s$ and $0^h\ 56^m\ 17^s$, respectively. From either pair of times, the total event time (from -70 to 137 km, the total time available for measurements) is 74 sec. The time spent in Earth shadow is $33^m\ 49^s$.

To apply these results, the following scheme can be used to determine the range of the mission parameters for, say, 1 year. The above calculation is repeated daily at 0 hours GMT. The orbital elements are updated each day by using the perturbation equations given in Brooks (1977) or Escobal (1965). If this is done for the example orbital elements given earlier, then such data parameters as shown in figures 8 through 11 can be derived. Figures 8, 9, 10, and 11 show as examples of typical output, the range of subtangent latitude, β, mission event time, and time in Earth shadow, respectively, plotted as functions of days from launch. Figures 10 and 11 show that mission event time ranges from about 70 to 100 seconds and that time in Earth shadow ranges from 29 to 35 minutes except when the orbit approaches a full sunlight situation (i.e., near days 60, 130, 240, and 310) and when β nears β_{max} (66° for this orbit) on days 170 and 350. At these times, the mission event times increase dramatically, and the potential for thermal problems in the instrument also increases. In addition, the time spent in Earth shadow decreases significantly, and there may be a reduction in the instrument recovery time or an adverse effect on the data transmission time.

Calculation of event nearest a specified geographic location.- As a second example of the application of these results to mission planning, suppose that we want to find an event which occurs nearest to some specified geographic location. For example, suppose a balloon launch or ground-based experiment is to be made during the sunrise or sunset event which occurs nearest to Laramie, Wyoming, whose coordinates are approximately 41° N and 105° W. From figure 8, it is seen that sunrise events occur at latitudes of 41° N at about 1 and 21 days after launch, and sunset events occur at 30 and 49 days after launch, with corresponding pairs of events occurring throughout the year. We want to find the conditions near the Laramie event at day 21. From a detailed output of the calculations described above, we find the following table of events for days 21 and 22:

Parameter	Day 21	Day 22
Date	November 2, 1985	November 3, 1985
Sunrise time	$0^h\ 48^m\ 06^s$	$1^h\ 00^m\ 27^s$
Subtangent point latitude	42.124° N	38.465° N
Subtangent point longitude	87.520° E	83.009° E

The total time between these events is 24.20583 hours. The orbital period is 360/0.06201 = 5805 seconds, or 1.6126 hours. Thus there are 24.20583/1.6126, or 15 events occurring during this interval. Since the longitude change is from east to west (opposite to the direction of the Earth's rotation), the total longitude spanned at 41° N during this time period is 360 + (87.520 - 83.009), or 364.511°, and thus the change in longitude per event is 364.511/15, or 24.300733 degrees per event. The change in longitude from 87.520° E to 105° W is about 192°, and thus the event nearest to Laramie will be the rounded-off quotient 192/24.300733, or the eighth event on November 2.

During these eight events, the actual longitude change is 194.40586°, so the event occurs at 106.886° W, or just west of Laramie.

The mean time between events is 24.20583/15 = 1.613722 hours, and hence the elapsed time for eight events is 12.909776 hours, or $12^h\ 54^m\ 35^s$. Thus, sunrise occurs at $12^h\ 54^m\ 35^s + 0^h\ 48^m\ 06^s$, or $13^h\ 42^m\ 41^s$ GMT.

The latitude of the subtangent point is also found by simple interpolation to be

$$42.124 + \frac{(38.465 - 42.124)}{15} \times 8 = 40.173° \text{ N}$$

A comparison between these interpolated results and a more detailed orbital calculation for this specific event gives the following table:

Parameter	This case	Exact
Time	$13^h\ 42^m\ 41^s$	$13^h\ 42^m\ 40^s$
Latitude	40.173° N	40.245° N
Longitude	106.886° W	106.858° W

For most ground-truth experiments tied to satellite rise or set events, the interpolated results appear to yield results of acceptable accuracy.

CONCLUDING REMARKS

An analytical method is developed for determining the geometrical parameters which are needed to describe the viewing angles of the Sun relative to an orbiting spacecraft when the Sun rises or sets with respect to the spacecraft. These equations are rigorous and are frequently used for parametric studies relative to mission planning and for determining instrument parameters.

The text is wholly self-contained in that no external reference to ephemerides or other astronomical tables is needed. Equations are presented which allow the computation of Greenwich sidereal time and right ascension and declination of the Sun generally to within a few seconds of arc, or a few tenths of a second in time.

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June 11, 1986

APPENDIX

COMPUTATION OF SOLAR POSITION

The following algorithm will produce the right ascension and declination of the Sun to within a few seconds of arc, as evidenced from a considerable amount of comparison with solar position data published in various annual almanacs. This appendix is completely self-contained. No other tables or reference materials such as almanacs or ephemerides are needed to compute the parameters discussed in the text; hence, these relations are quite suitable for computer use. Angles are computed mod (360). The steps and equations of the algorithm are

1. Compute the Julian date (Almanac for Computers 1980)

$$JD = 367K - \left\langle \frac{7[K + \langle (M + 9)/12 \rangle]}{4} \right\rangle + \left\langle \frac{275M}{9} \right\rangle + I + 1721013.5 + \frac{GMT}{24}$$

where

JD	Julian date
K	year (e.g., 1984)
M	month ($1 \leq M \leq 12$)
I	day of month ($1 \leq I \leq 31$)
GMT	Greenwich mean time, hours ($0 \leq GMT \leq 23.99$)

The symbol $\langle\ \rangle$ denotes the maximum integer value.

2. Compute the sidereal time at Greenwich at 0 hours GMT of a given date as follows (Escobal 1965):

$$STGO = 99.6909833° + 36000.7689° T_u + 0.00038708° T_u^2$$

where T_u is defined by

$$T_u = \frac{JD - 2\ 415\ 020}{36\ 525}$$

The equation for STGO includes the precession of the equinoxes but neglects the nutation terms, that is, the equation of the equinoxes. Hence, this is the mean sidereal time at 0 hours GMT. For any GMT, the sidereal time is then found from

$$STG = STGO + 0.25068447 GMT$$

where GMT is in minutes.

3. Compute the right ascension of the mean Sun (RAMS) (Escobal 1965)

$$\text{RAMS} = 279.6966778° + 36000.76892° T_u + 0.0003025° T_u^2$$

4. Compute the obliquity of the ecliptic ε (Escobal 1968)

$$\varepsilon = 23.45229444° - 0.0130125° T_u - 0.0000016389° T_u^2$$

5. Compute the Sun's apparent mean anomaly M_s (Escobal 1968)

$$M_s = 358.475844° + 35999.04975° T_u - 0.00015° T_u^2 - 0.0000033333° T_u^3$$

6. Compute the eccentricity of the Earth's orbit e_e (Escobal 1968)

$$e_e = 0.01675104 - 0.00004180 T_u - 0.000000126 T_u^2$$

7. Compute the equation of time E_T (Smart 1977)

$$E_T = y \sin 2\ell - 2e_e \sin M_s + 4e_e y \sin M_s \cos 2\ell$$

$$- \frac{1}{2} y^2 \sin 4\ell - \frac{5}{4} e_e^2 \sin 2M_s + \Big[\frac{5}{2} y e_e^2 \sin 2M_s \cos 2\ell$$

$$- 4y^2 e_e \sin M_s \cos 4\ell + \frac{1}{3} y^3 \sin 6\ell + \frac{1}{4} e_e^3 \sin M_s$$

$$- \frac{13}{12} e_e^3 \sin 3M_s \Big] + \ldots$$

where

$$\ell = \text{RAMS}$$

$$y = \tan^2 \frac{\varepsilon}{2}$$

The term in brackets is of third order in the Earth eccentricity and may be omitted unless extreme accuracy is required.

If one substitutes the values of y and e_e for mid-1985, the following equation (Smart 1977) will give reasonably accurate numerical values for the decade of the 1980's:

$$E_T = -0.4329 \sin \ell - 1.7900 \cos \ell + 2.4848 \sin 2\ell$$
$$- 0.0083 \cos 2\ell + 0.0179 \sin 3\ell + 0.0804 \cos 3\ell$$
$$- 0.0529 \sin 4\ell + \ldots$$

The units are degrees.

8. Compute the right ascension of the true Sun α_s (Smart 1977)

$$\alpha_s = \mathrm{RAMS} - E_T$$

9. Compute the declination of the Sun δ_s

$$\tan \delta_s = \sin \alpha_s \tan \varepsilon$$

10. If needed, compute the hour angle of the mean Sun (HAMS) and the hour angle of the true Sun (HAS) (Smart 1977)

$$\mathrm{HAMS} = \mathrm{GMT} - 12^h - \mathrm{Longitude}$$

$$\mathrm{HAS} = \mathrm{HAMS} + E_T$$

Longitude is defined as positive east of Greenwich. The west longitude of the true Sun is just the hour angle of the true Sun measured at Greenwich. The latitude of the true Sun is, of course, identical with the declination.

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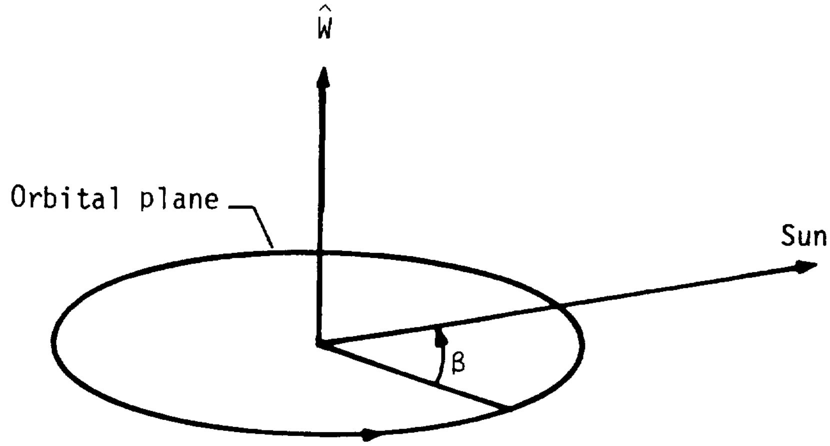

Figure 1.- Definition of the angle β. $\hat{W}$ is a unit vector normal to the orbital plane.

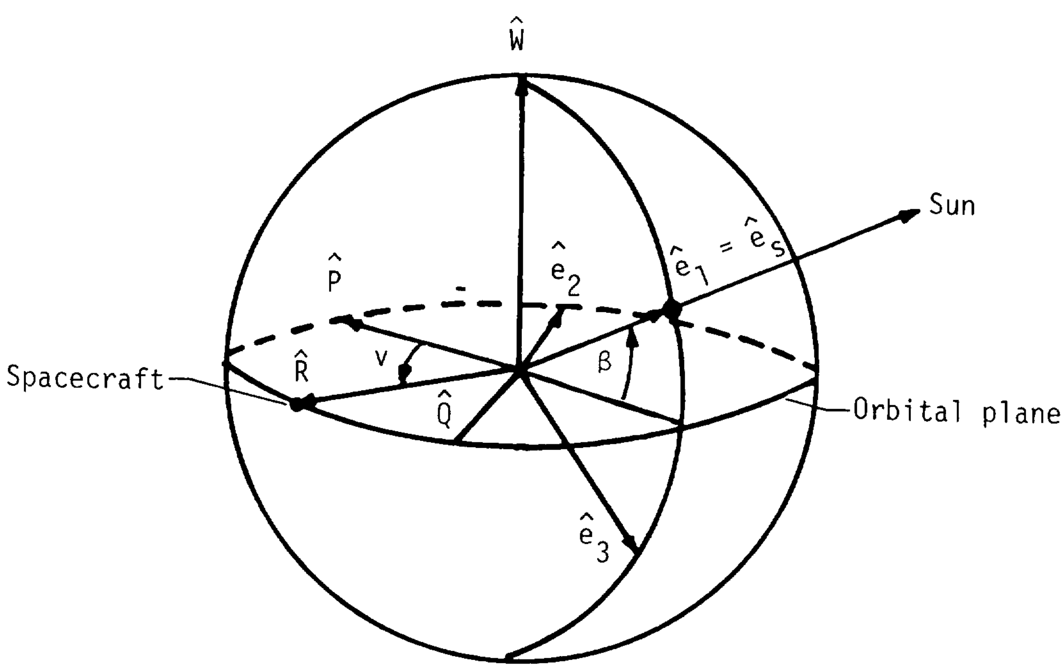

Figure 2.- Projection geometry and definitions of unit vectors $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$.

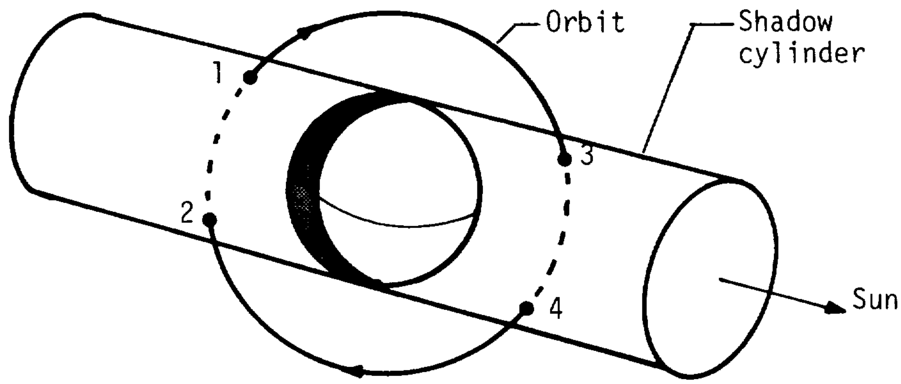

Figure 3.- Shadow cylinder and orbit entry and exit points.

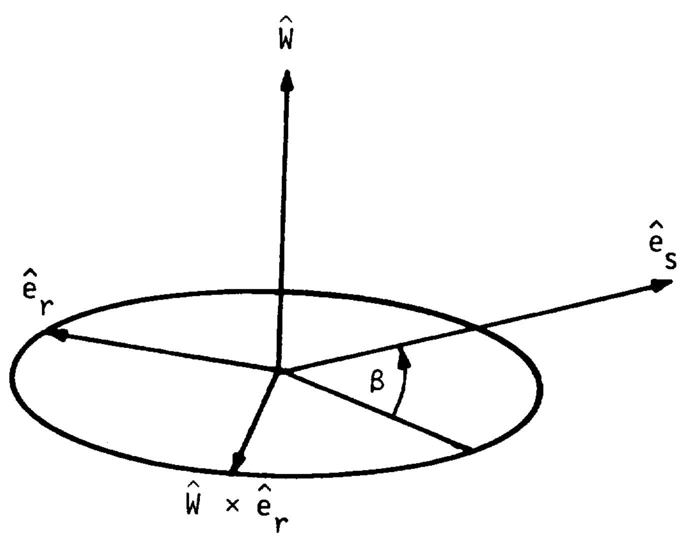

Figure 4.- Local spacecraft coordinate system.

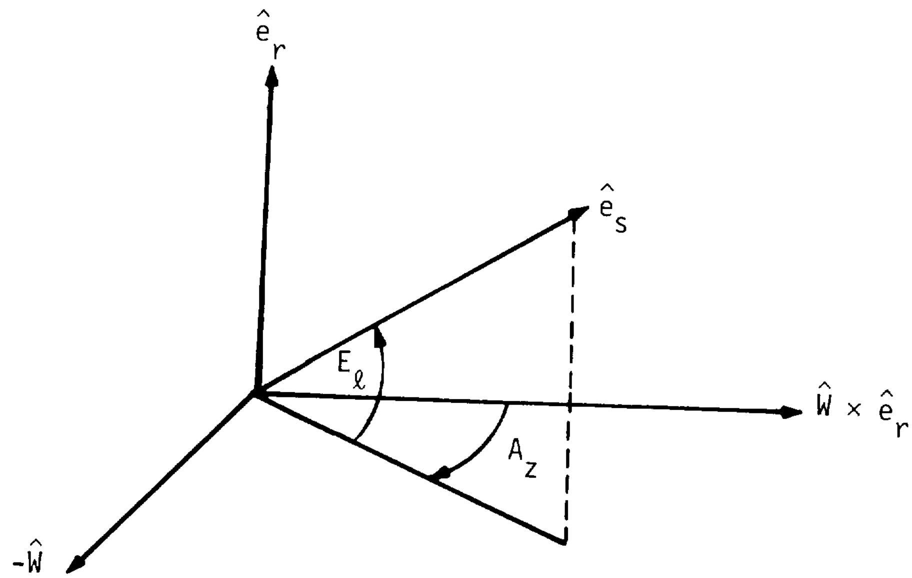

Figure 5.- Definition of azimuth and elevation angles in local spacecraft coordinate system.

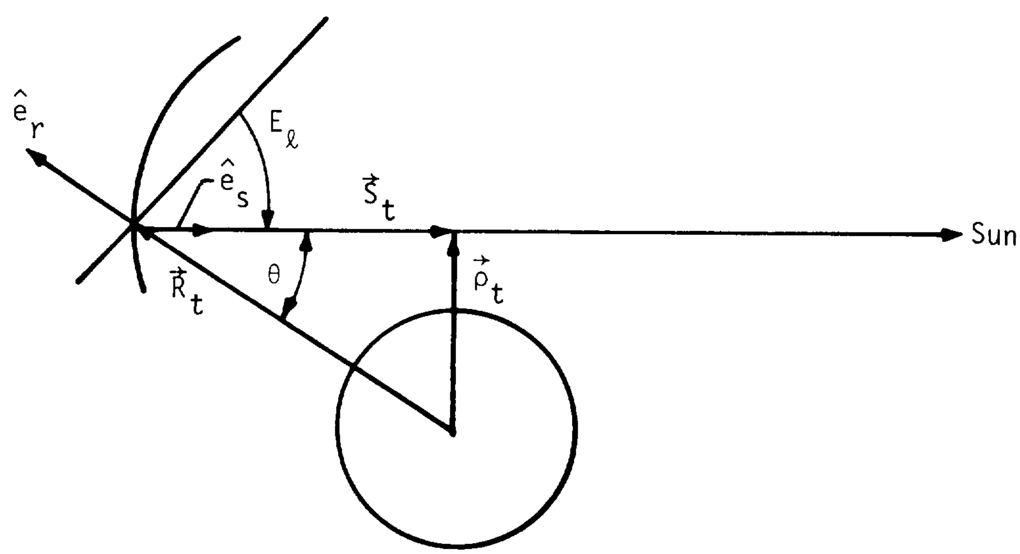

Figure 6.- Sketch showing tangent point geometry.

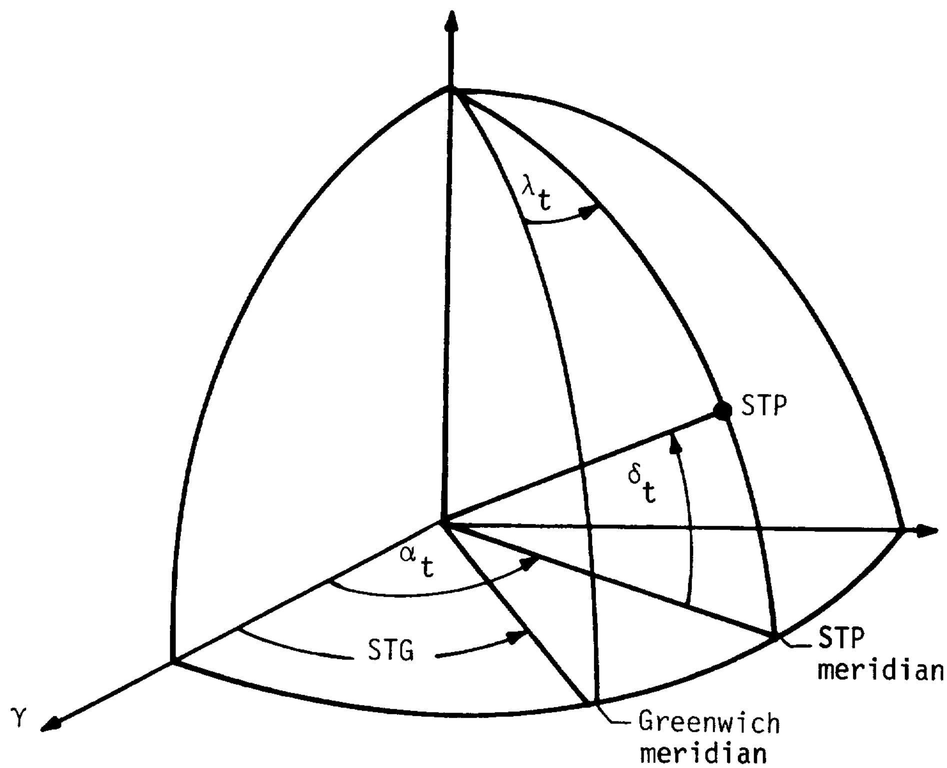

Figure 7.- Relationship between Greenwich meridian, longitude, and sidereal time at Greenwich. STP is the subtangent point.

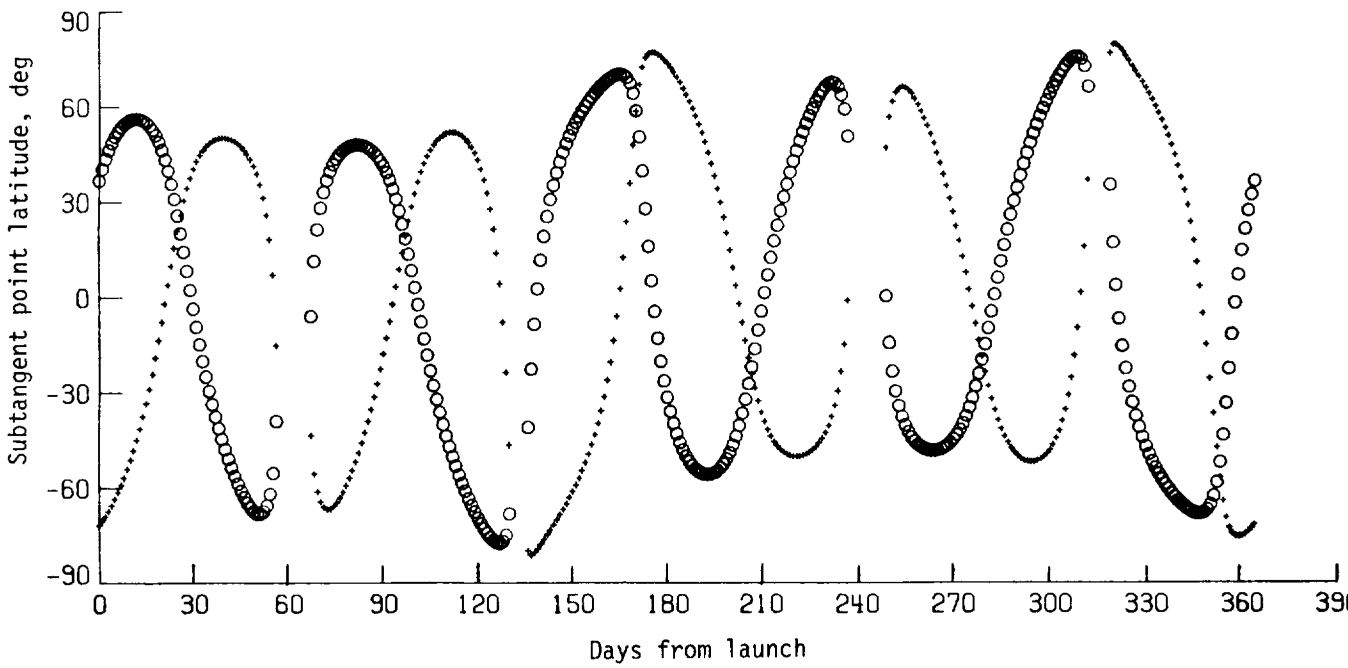

Figure 8.- Annual variation of subtangent point latitude for example orbit used in text.

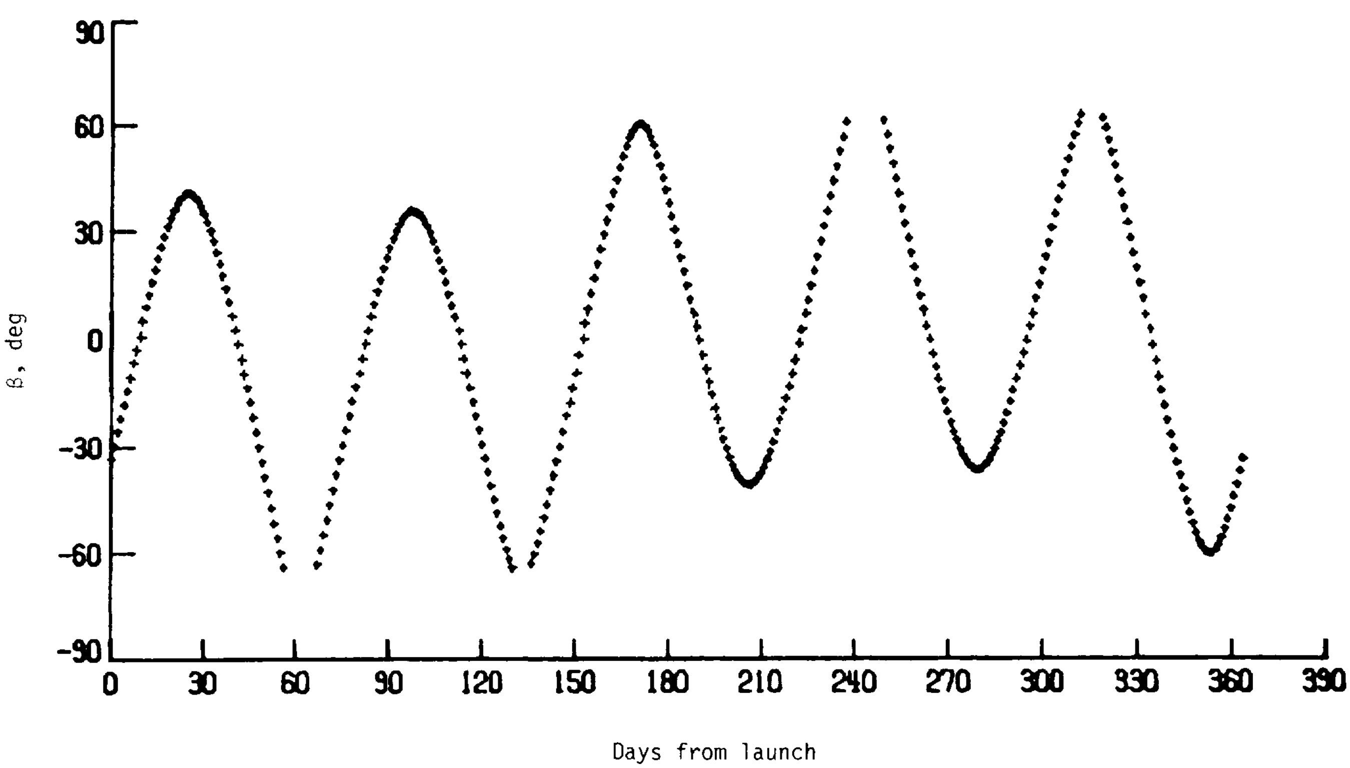

Figure 9.- Annual variation of β for example orbit.

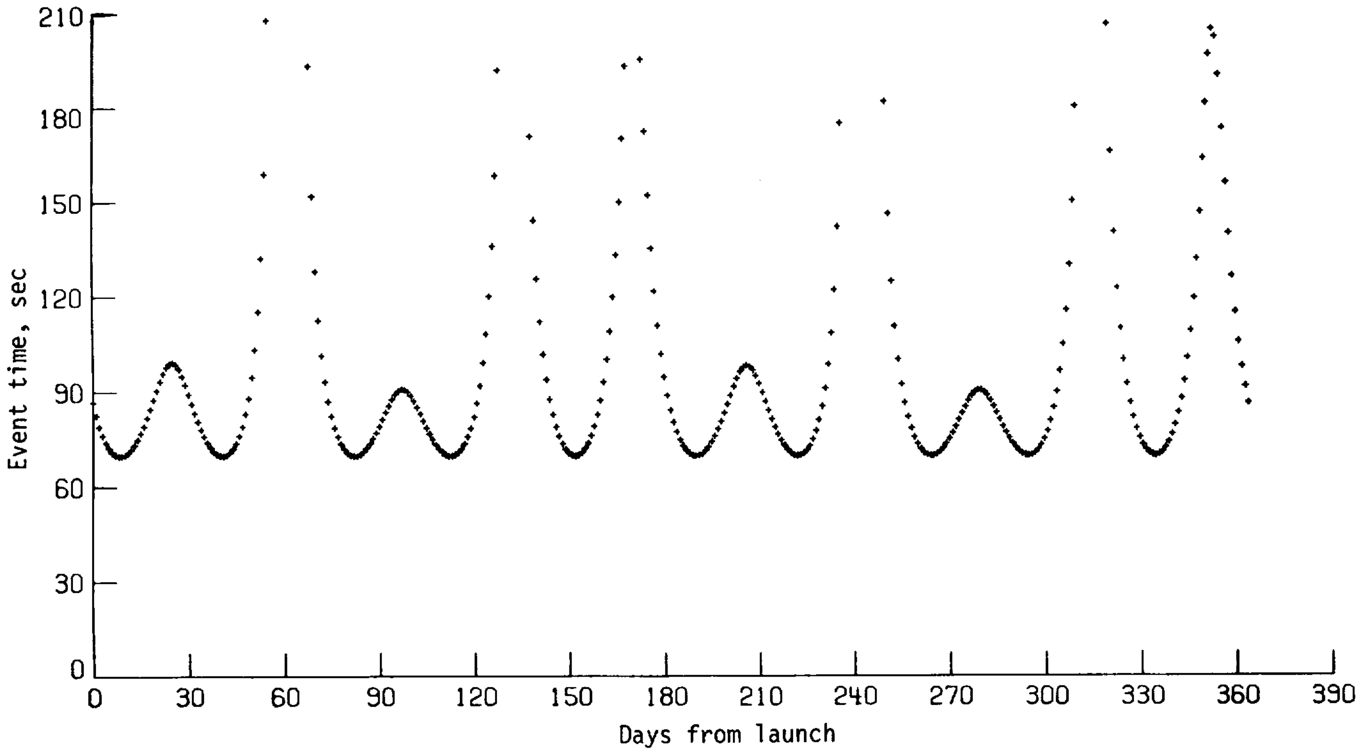

Figure 10.- Annual variation of event duration for example orbit.

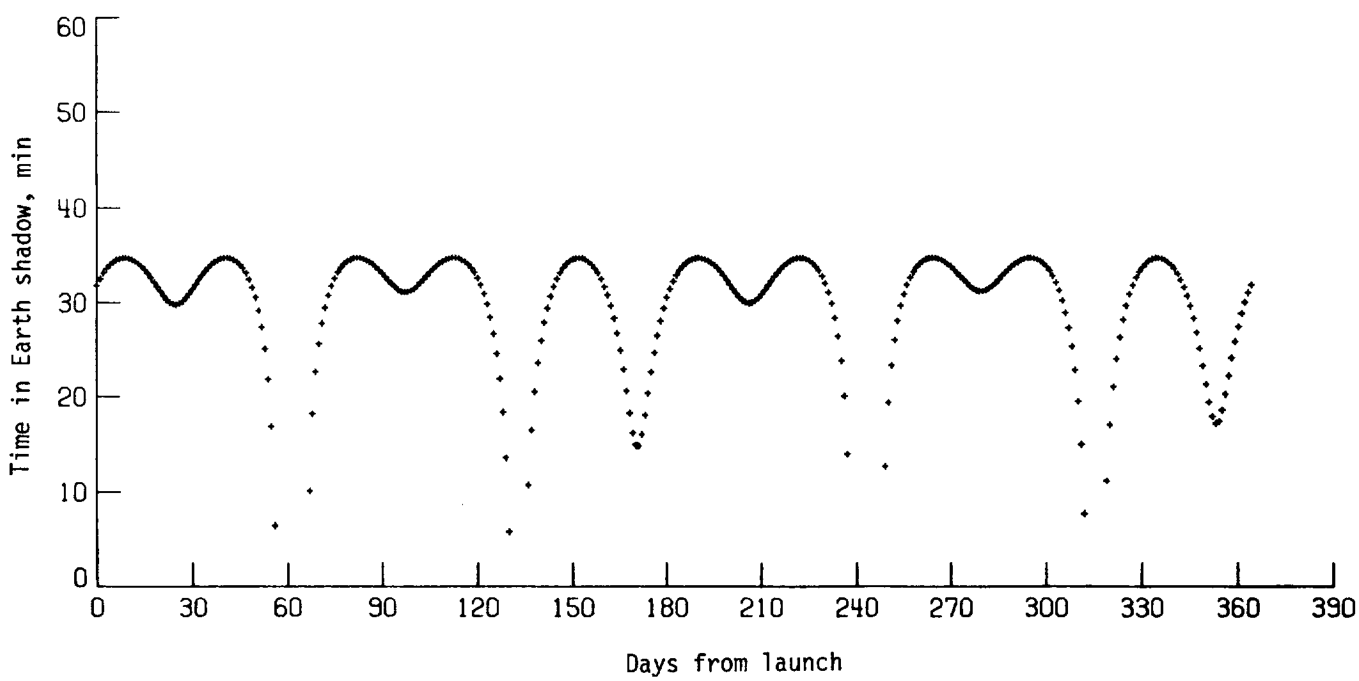

Figure 11.- Annual variation of time in Earth's shadow for example orbit.

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16. Abstract

An analytical method is developed for determining the geometrical parameters which are needed to describe the viewing angles of the Sun relative to an orbiting spacecraft when the Sun rises or sets with respect to the spacecraft. These equations are rigorous and are frequently used for parametric studies relative to mission planning and for determining instrument parameters. The text is wholly self-contained in that no external reference to ephemerides or other astronomical tables is needed. Equations are presented which allow the computation of Greenwich sidereal time and right ascension and declination of the Sun generally to within a few seconds of arc, or a few tenths of a second in time.

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